

# THE ELECTROMAGNETIC COUPLING IN KEMMER-DUFFIN-PETIAU THEORY

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## Abstract

We analyse the electromagnetic coupling in the Kemmer-Duffin-Petiau (KDP) equation. Since the KDP-equation which describes spin-0 and spin-1 bosons is of Dirac-type, we examine some analogies and differences from the Dirac equation. The main difference to the Dirac equation is that the KDP equation contains redundant components. We will show that as a result certain interaction terms in the Hamilton form of the KDP equation do not have a physical meaning and will not affect the calculation of physical observables. We point out that a second order KDP equation derived by Kemmer as an analogy to the second order Dirac equation is of limited physical applicability as (i) it belongs to a class of second order equations which can be derived from the original KDP equation and (ii) it lacks a back-transformation which would allow one to obtain solutions of the KDP equation out of solutions of the second order equation. We therefore suggest a different higher order equation which, as far as the solutions for the wave functions are concerned, is equivalent to the original first order KDP wave equation.

# 1 Introduction

Since the early days of Quantum Mechanics physicists have tried to construct relativistically invariant wave equations for different spins and in different disguises. Beside the, by now standard, Klein-Gordon and Dirac equations there are other Lorentz invariant wave equations. We mention here the infinite component equations of Majorana type which describe a tower of higher spin states [1], an equation suggested by Dirac for spin-0 bosons [2] and its generalizations [3], the six dimensional Weinberg–Shay–Good equation for spin-1 bosons [4] and the Kemmer equation, also called Kemmer-Duffin-Petiau equation [5], for spin-0 and spin-1 bosons. It is the last set of equations which, in the presence of an electromagnetic field, we will investigate here more closely. Although we have a consistent description of bosons in quantum field theory in terms of the Klein–Gordon equation it is often instructive and useful to look at the same problem from a yet different point of view, in the case of concern here from the perspective of the KDP equation. Indeed, as we will sketch below, new insights can still be gained. The KDP equation which has been known for some time is an equation of Dirac type

$$(i\beta_\mu\partial^\mu + m)\psi = 0 \quad (1.1)$$

where the matrices  $\beta_\mu$  satisfy a trilinear algebra [6]

$$\beta^\mu\beta^\nu\beta^\alpha + \beta^\alpha\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\alpha + g^{\nu\alpha}\beta^\mu \quad (1.2)$$

The convention for the metric tensor is here  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . The algebra (1.2) has three distinct representations: the trivial one with  $\beta_\mu = 0$ , a five-dimensional representation describing spin-0 particles and a ten dimensional one for the spin-1 bosons. One can derive (1.1) from a Lagrangian, construct interaction Lagrangians with other fields and quantize the theory. An equation similar to (1.1) can be also written down for the massless case. We will not dwell here on these issues and refer the interested reader to [7] and [8] where these topics are treated in more details.

The renewed interest in (1.1) can be attributed to the discovery of a new conserved four-vector current whose zeroth component is positive definite and can therefore be interpreted as a probability density to find a particle at a point  $r$  [9], [10], [11]. Strictly speaking from (1.1) one can construct two conserved four-vector currents. The first one

$$\begin{aligned} j^\mu &\equiv \bar{\psi}\beta^\mu\psi \\ \partial_\mu j^\mu &= 0 \end{aligned} \quad (1.3)$$

with  $\bar{\psi} = \psi^\dagger(2\beta_0^2 - 1)$  shares with Klein-Gordon theory the property that  $j_0$  is not positive definite. Defining

$$\phi \equiv \frac{\psi}{\sqrt{\int d^3x \psi^\dagger \psi}} \quad (1.4)$$

one can also derive from (1.1) a different conservation law in the form

$$\partial_\mu s^\mu \equiv \partial_0(\phi^\dagger \phi) + \partial_i(\phi^\dagger \tilde{\beta}_i \phi) = 0 \quad (1.5)$$

where  $\tilde{\beta}_i = [\beta_0, \beta_i]$ . Based on an idea of Holland [12] to use the symmetric energy-momentum tensor  $\theta_{\mu\nu}$  to construct conserved four-vectors Ghose, Home and Roy were able to give a proof that  $s^\mu = (s^0, s^i)$  indeed transforms as a four vector [9, 10, 11]. In contrast to (1.3) we have now clearly  $s^0 > 0$ . It would be probably tedious, if not impossible, to establish such a result in the Klein-Gordon theory. In this letter we will not examine further consequences of the current (1.5), but would only like to point out that such a new current is welcome and useful especially for neutral particles. For instance, in [13] a non-relativistic current for neutral kaons has been used to compute space (in contrast to time) dependent conversion probabilities  $P_{K^0 \rightarrow \bar{K}^0}(r)$  and  $P_{\bar{K}^0 \rightarrow K^0}(r)$  by integrating the current over time and surface through which the particles pass. With the help of (1.5) the same exercise could be now done for a general relativistic current [14]. This is not a mere academic problem as the ongoing discussion about the interference phase appearing in the conversion probabilities shows (for details see [13]).

Another interesting aspect which will be the main subject of the present paper is the external field problem in connection with (1.1). An external field problem in the context of a relativistic equation is only well defined provided the external field is not too strong. But once we restrict ourselves to weak fields, useful results for energy eigenvalues and for wave functions can be obtained. This is the case with the Dirac equation in a homogeneous magnetic field and a plane wave electromagnetic field (see e.g. [15]).

The interest in the external field problem in the KDP equation is twofold. First, since (1.1) looks formally like the Dirac equation, it is useful for practical purposes (like methods to solve the KDP equation, etc) to see how far one can stretch the analogy to the Dirac equation. Note that because of the existence of the current (1.5) a relativistic quantum mechanics of bosons is now possible. Secondly and more importantly, it is not a priori obvious if the KDP equation including interactions is equivalent to the Klein-Gordon description (it is easy to show that in the free case both theories are equivalent)\*. When we employ the minimal coupling scheme for the electromagnetic four-potential  $A_\mu$  in eq. (1.1) and go over to the Hamilton formulation of the theory we will discover a term which seems to lack a physical interpretation. Kemmer has discussed a similar looking term in a second order Klein-Gordon type of equation which in analogy to second order Dirac equation (see [15]) he derived from (1.1). This has led to a suggestion to couple the electromagnetic field only at the level of the Hamilton form which would then avoid the cumbersome term [11]. It will be shown, however, that this cumbersome term has no physical

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\*It might be also worthwhile to remark that in view of (1.1) certain mixing phenomena, like  $\omega^0 - \rho^0$  and  $\eta - \eta'$  mixing, when formulated in the Lagrangian language, could be now linear, as opposed to quadratic, in the mass parameters

meaning and it vanishes when the first order wave equation is reduced to its physical components. Moreover, the analogy of the second order Kemmer equation to the similar looking second order Dirac equation is very limited. Whereas in the latter case we can transform solutions of the second order equation to solutions of the Dirac equation and vice versa, such a one-to-one correspondence is not possible in the Kemmer case. More seriously the second order Kemmer equation is only one member of a class of second order equations which, in principle, can be derived from (1.1). Their physical significance is therefore not clear. On account of this we suggest a higher order wave equation which has the same virtue as the second order Dirac equation (i.e. a back-transformation to the solutions of the first order equation), but does not, in the free case, reduce to the Klein-Gordon equation. This in turn implies that we should not expect that every term in such an equation has an interpretable meaning.

## 2 Algebra

Before discussing the external field problem it is convenient for calculational as well as physical reasons to separate the spin-0 case from the spin-1 case by purely algebraic means. In [7] it was shown that given the algebra (1.2) there exists a matrix  $\beta$  satisfying

$$\begin{aligned}\beta^2 &= \beta \\ \{\beta, \beta_\mu\} &= \beta_\mu\end{aligned}\tag{2.1}$$

Obviously  $\beta' = 1 - \beta$  will also satisfy (2.1). For the spin-0 case it is then possible to give a relation which is stronger than (1.2), namely [7]

$$\beta^\mu \beta^\nu \beta^\alpha = g^{\mu\nu} \beta^\alpha \beta + g^{\nu\alpha} \beta \beta^\mu\tag{2.2}$$

It is easy to show that (1.2) follows from (2.2). We supplement this remark by observing that the spin-1 algebra can be also characterized by a algebraic relation different from (1.2). Defining the matrix  $\omega$  by

$$\omega \equiv \frac{i}{4} \epsilon^{\mu\nu\sigma\delta} \beta_\mu \beta_\nu \beta_\sigma \beta_\delta\tag{2.3}$$

( $\epsilon^{0123} = 1$ ) we see that  $\omega = 0$  for spin-0 on account of (2.2). One can deduce then several properties of  $\omega$  [7] out of which we quote

$$\omega^2 = 1 - \beta\tag{2.4}$$

$$\{\omega^2, \beta_\mu\} = \beta_\mu\tag{2.5}$$

$$\beta_\mu \omega \beta_\nu + \beta_\nu \omega \beta_\mu = 0\tag{2.6}$$

$$\beta_\mu \beta_\nu \omega + \omega \beta_\nu \beta_\mu = g_{\mu\nu} \omega\tag{2.7}$$

The matrix  $\omega$  has a close analogy to the  $\gamma_5$  matrix in the Clifford algebra and can be used to extend (1.2) to five dimensional space-time. Using now (2.5)-(2.7) we get

$$\begin{aligned}\beta^\nu \beta^\alpha \beta^\mu &= g^{\alpha\mu} \beta^\nu \omega^2 + g^{\nu\alpha} \omega^2 \beta^\mu \\ &+ \beta^\mu \omega \beta^\nu \beta^\alpha \omega + \omega \beta^\alpha \beta^\mu \omega \beta^\nu\end{aligned}\quad (2.8)$$

With the help of eq. (2.7) it is not difficult to see that (2.8) leads again to the original algebra (1.2). Hence we can state that the  $(\beta, \beta_\mu)$  algebra (2.2) characterizes the matrices for the spin-0 case whereas the  $(\omega, \beta_\mu)$  algebra does the same for the spin-1 case.

### 3 The wave equations in an external field

In the following we will use the minimal electromagnetic coupling scheme with a free electromagnetic field,  $\partial_\mu F^{\mu\nu} = 0$ . The covariant derivative and the corresponding operator are then defined as usual by

$$\begin{aligned}\mathcal{D}^\mu &\equiv i\partial^\mu - eA^\mu \\ \Lambda(\mathcal{D}) &\equiv \beta_\mu \mathcal{D}^\mu + m\end{aligned}\quad (3.1)$$

so that the wave equation in the presence of an electromagnetic field reads now

$$\Lambda(\mathcal{D})\psi = 0 \quad (3.2)$$

Below we will give a brief derivation of three equations which are important in the KDP theory. For more details we refer the reader to [5]. These three equations will be the basis for our discussion of the external field problem in the KDP theory.

By multiplying (3.2) with  $\mathcal{D}^\rho \beta_\rho \beta_\nu$  we obtain

$$\mathcal{D}_\nu \psi = \beta_\rho \beta_\nu \mathcal{D}^\rho \psi + \frac{ie}{2m} F^{\mu\rho} (\beta_\rho \beta_\nu \beta_\mu + \beta_\rho g_{\mu\nu}) \psi \quad (3.3)$$

where the standard result  $[\mathcal{D}^\mu, \mathcal{D}^\rho] = ieF^{\rho\mu}$  has been used. Combining (3.2) with (3.3) one can cast the KDP equation into a Hamilton form

$$\begin{aligned}i\partial_0 \psi &= H\psi \\ H &= \mathcal{D}^i [\beta_i, \beta_0] - \beta_0 m + eA_0 + \frac{ie}{2m} F^{\mu\rho} (\beta_\rho \beta_0 \beta_\mu + \beta_\rho g_{\mu 0})\end{aligned}\quad (3.4)$$

Finally one can also derive from (3.2) and (3.3) the second order equation mentioned in the introduction. It reads [5]

$$\Omega_1(\mathcal{D})\psi \equiv \left[ \mathcal{D}^\alpha \mathcal{D}_\alpha - m^2 - \frac{ie}{2} F^{\nu\mu} S_{\nu\mu} - \frac{ie}{2m} (\beta_\rho \beta_\nu \beta_\mu + \beta_\rho g_{\mu\nu}) \mathcal{D}^\nu F^{\mu\rho} \right] \psi = 0 \quad (3.5)$$

where

$$S_{\nu\mu} = \beta_\nu\beta_\mu - \beta_\mu\beta_\nu \quad (3.6)$$

whose spatial components,  $S_{ij}$ , are related to the spin operators. A few comments are in order now.

- (i) The troublesome term referred to earlier appears in all three equations, (3.3), (3.4) and (3.5) in different disguises. It is proportional to  $ie/2m$  and is always connected with the tensor matrix

$$(\beta_\rho\beta_\nu\beta_\mu + \beta_\rho g_{\mu\nu})$$

Kemmer [5] compared (3.5) with the second order Dirac equation (see section 5) and pointed out that a term corresponding to  $F^{\mu\nu}S_{\mu\nu}$  appears also in the latter equation, but not the other lengthy term involving one and three  $\beta_\mu$  matrices. Equation (3.5) consists of the Klein-Gordon operator plus terms which one could try to interpret as spin-field interaction [5] (this is certainly true for  $F^{\mu\nu}S_{\mu\nu}$ , see, however, the points (ii) and (iii) below for the rest of the terms). It is then tempting to view (3.5) as a close analogue to the second order Dirac equation. We will address this question in section 5 in more detail and show that this is not the case.

- (ii) Note that in (3.5) the derivative acts on the wave function as well as on the electromagnetic stress tensor  $F^{\mu\nu}$ . However, eq. (2.2) tells us that for the spin-0 case we have

$$(\beta_\rho\beta_\nu\beta_\mu + \beta_\rho g_{\mu\nu})(\partial^\nu F^{\mu\rho}) = 0$$

This is not the case for spin-1 which immediately gives rise to the question about the meaning of the derivatives of the electric and magnetic field in eq. (3.5). Such derivatives acting on the physical electromagnetic fields do of course enter the Maxwell equations. However, we are faced here with the external field problem where the electromagnetic field configurations are supposed to be given.

- (iii) If we look at the photon interaction with bosons from the point of view of Feynman diagrams then, already at tree level, there will be two contributions: a one-photon exchange with derivative couplings and a two-photon contact diagram at the level  $e^2$ . We would therefore expect in (3.5) a spin-field interaction term at the order  $e^2$ . Indeed a part of the terms in (3.5) can be rewritten in such a way as to confirm this expectation. Using once more eq. (3.3) we can write

$$\begin{aligned} & \frac{ie}{2m}(\beta_\rho\beta_\nu\beta_\mu - \beta_\rho g_{\mu\nu})F^{\mu\rho}\mathcal{D}^\nu\psi \\ &= \frac{e^2}{4m^2}F^{\alpha\gamma}F^{\mu\rho}(\beta_\rho\beta_\gamma\beta_\mu\beta_\alpha + \beta_\rho\beta_\gamma g_{\mu\alpha})\psi \\ &= -\frac{e^2}{16m^2}\left[F^{\alpha\gamma}F^{\mu\rho}S_{\rho\mu}S_{\alpha\gamma} - 2F^{\alpha\gamma}F^{\mu\rho}S_{\rho\gamma}S_{\mu\alpha}\right]\psi + \frac{e^2}{2m^2}F^{\alpha\gamma}F_{\alpha}^{\rho}\beta_{\rho}\beta_{\gamma}\psi \quad (3.7) \end{aligned}$$

It is a matter of the relative sign in  $(\beta_\rho\beta_\nu\beta_\mu + \beta_\rho g_{\mu\nu})$  and the derivatives acting on  $F^{\mu\nu}$  that we cannot attribute the whole term  $(\beta_\rho\beta_\nu\beta_\mu + \beta_\rho g_{\mu\nu})\mathcal{D}^\nu F^{\mu\rho}\psi$  to spin-field interaction at the order  $e^2$ .

- (iv) Eq. (3.5) follows from (3.2), but not vice versa. In other words, any solution of (3.2) will be also a solution of (3.5), but the opposite is not necessarily true. To make eq. (3.5) meaningful one would need a prescription which would convert solutions of (3.5) to solutions of the KDP equation (3.2). We will address this issue in more detail in section 5.
- (v) Finally, had we first taken a free Hamilton operator and coupled to it, via the minimal substitution, the electromagnetic field then the term in (3.4) proportional to  $ie/2m$  would be absent [11]. Two possibilities can occur now. Either the Lagrangian method and the Hamiltonian one to couple the electromagnetic field are indeed different or the term proportional  $ie/2m$  is unphysical.

Since the wave function in (3.2) contains redundant components an important part of the KDP theory are the constraints. They are given by

$$\mathcal{C}[\psi] \equiv \beta_i\beta_0^2\mathcal{D}^i\psi + m(1 - \beta_0^2)\psi = 0 \quad (3.8)$$

Eq. (3.8) is obtained from (3.2) by multiplying the latter with  $(1 - \beta_0^2)$  and using the identity  $(1 - \beta_0^2)\beta_i = \beta_i\beta_0^2$  which follows from (1.2).

At this stage it is quite straightforward to investigate more closely the spin-0 case. The problem of spin-1 will be discussed in the next section. An explicit representation of the  $\beta_\mu$  matrices for the spin-0 case reveals that (3.8) takes the form

$$\mathcal{D}^i\psi + m\psi_i = 0, \quad i = 1, 2, 3 \quad (3.9)$$

We can also eliminate  $\psi_4$  by

$$\mathcal{D}^0\psi_5 + im\psi_4 = 0 \quad (3.10)$$

which is a simple consequence of (3.2). In the spin-0 case we arrive then at the standard result for the single physical component  $\psi_5$

$$(\mathcal{D}^\alpha\mathcal{D}_\alpha - m^2)\psi_5 = 0 \quad (3.11)$$

Despite the fact that each of the eqs. (3.3-3.5) constitutes a complicated system of coupled partial differential equations, being of course also valid for spin-0, the end result in terms of the physical component is quite simple. This signals that not every term in the eqs. (3.3-3.5) has a physical relevance. Below we will perform a similar reduction to the physical components of the spin-1 wave function and obtain an equation which determines these components.

## 4 The reduction of the wave equation

In the massive spin-1 case there are three physical components which in the representation we are using <sup>†</sup> are given by  $(\psi_1, \psi_2, \psi_3)$ . The unphysical variables  $(\psi_7, \psi_8, \psi_9)$  and  $\psi_{10}$  can be eliminated by using

$$\begin{aligned} \mathcal{D}^0 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} - im \begin{pmatrix} \psi_7 \\ \psi_8 \\ \psi_9 \end{pmatrix} &= 0 \\ \sum_{i=1}^3 \mathcal{D}^i \psi_i + im \psi_{10} &= 0 \end{aligned} \quad (4.1)$$

The first equation in (4.1) follows directly from the wave equation (3.2). The second one is part of the constraints (3.8). The other elements  $(\psi_4, \psi_5, \psi_6)$  are related to  $(\psi_7, \psi_8, \psi_9)$  also through the constraints (3.8). For our purposes it suffices, however, to reduce the eq. (3.2) or equivalently (3.4) to a  $6 \times 6$  equation determining the wave function  $(\psi_i, \psi_j)$ ,  $i=1,2,3$  and  $j=7,8,9$ . This is exactly the wave function given by  $\beta_0^2 \psi$ . The reduction process of eq. (3.2) or alternatively (3.4) is essentially based on the idea of incorporating the constraints (3.8) into the wave equation. Choosing (3.4) this yields then the formula

$$\begin{aligned} i\partial_0(\beta_0^2 \psi) &= \left[ \beta_0^2 H \beta_0^2 - \frac{1}{m} \beta_0^2 H \beta_i \mathcal{D}^i \beta_0^2 \right] (\beta_0^2 \psi) \\ &\equiv \mathcal{O}_{red}(\beta_0^2 \psi) \end{aligned} \quad (4.2)$$

which should be regarded simply as an equation to determine the reduced components  $\beta_0^2 \psi$ . We do not identify  $\mathcal{O}_{red}$  with a Hamiltonian, because the operator  $\mathcal{O}_{red}$  is not hermitian. A more compact form of the new operator is obtained by noticing that

$$\begin{aligned} \beta_0^2(\beta_{i_1} \beta_{i_2} \dots \beta_{i_{2n+1}}) \beta_0^2 &= 0 \\ \beta_0^2(\beta_{i_1} \beta_{i_2} \dots \beta_{i_{2n}}) &= (\beta_{i_1} \beta_{i_2} \dots \beta_{i_{2n}}) \beta_0^2 = \beta_0^2(\beta_{i_1} \beta_{i_2} \dots \beta_{i_{2n}}) \beta_0^2 \end{aligned} \quad (4.3)$$

$\mathcal{O}_{red}$  can be then put in the following form

$$\mathcal{O}_{red} = -\beta_0 m + \beta_0^2 e A_0 + \frac{1}{m} \beta_0 \beta_i \beta_j \mathcal{D}^i \mathcal{D}^j \quad (4.4)$$

Since the  $\beta_i$  matrices do not have a direct interpretation as operators in the Hilbert space it would be desirable at this point to make some contact with spin operators  $S_k$ . We can do this by defining  $S_k$  through

$$S_{ij} = -i\epsilon_{ijk} S_k \quad (4.5)$$

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<sup>†</sup>We are using here the  $\beta_0, i\beta_i$  where the  $\beta_i$  can be found in [5]



Indeed in our explicit representation, (4.6) implies that

$$S_k = \begin{pmatrix} T_k & 0_{3 \times 3} & 0_{3 \times 3} & \bar{0} \\ 0_{3 \times 3} & T_k & 0_{3 \times 3} & \bar{0} \\ 0_{3 \times 3} & 0_{3 \times 3} & T_k & \bar{0} \\ \bar{0}^T & \bar{0}^T & \bar{0}^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} T_k & 0_{3 \times 3} \\ 0_{3 \times 3} & T_k \end{pmatrix} \quad (4.6)$$

where  $(T_k)_{ij} = i\epsilon_{ikj}$  is the 3-dimensional representation of  $SU(2)$  and  $\bar{0}$  is a 3-dimensional zero vector  $\bar{0}^T = (0, 0, 0)$ . As a result, the correct commutation relation will emerge

$$[S_i, S_j] = i\epsilon_{ijk} S_k \quad (4.7)$$

Since  $\beta_0^2$  projects onto a six dimensional space we have displayed in (4.6) the form of  $S_k$  to which it will reduce in this space. Besides the commutation relation (4.7) it is convenient to have yet another, preferably anti-commutative relation which would relate the  $\beta_i$  to the spin operators  $S_k$ . One can find such a relation by introducing the auxiliary matrix

$$\xi = \begin{pmatrix} 1_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & \bar{0} \\ 0_{3 \times 3} & -1_{3 \times 3} & 0_{3 \times 3} & \bar{0} \\ 0_{3 \times 3} & 0_{3 \times 3} & -1_{3 \times 3} & \bar{0} \\ \bar{0}^T & \bar{0}^T & \bar{0}^T & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & -1_{3 \times 3} \end{pmatrix} \quad (4.8)$$

where again the  $6 \times 6$  expression of this matrix has been displayed. By explicit calculation one can check that the identities

$$\begin{aligned} \{\beta_i, \beta_j\} &= \xi \{S_i, S_j\}, \quad i \neq j \\ \beta_k^2 &= -\frac{1}{2}(1 + \xi) + \xi S_k^2 \end{aligned} \quad (4.9)$$

hold. Another useful formula involving the spin matrices follows simply from the fact that we are working with the adjoint representation  $(T_k)_{ij} = i\epsilon_{ikj}$ . It is trivial to see that the  $T_k$ , and hence the  $S_k$ , fulfill an algebra similar to (1.2), namely

$$S_i S_k S_j + S_j S_k S_i = \delta_{ik} S_j + \delta_{jk} S_i \quad (4.10)$$

The Duffin algebra (1.2) is essentially a covariant generalization of (4.10). Having introduced the spin operators we can rewrite (4.4) into a physically more suitable form

$$\mathcal{O}_{red} = -\beta_0 m + e A_0 + \frac{1}{2m} \beta_0 (1 + \xi) (\mathcal{D}^i \mathcal{D}_i) + \frac{1}{m} \beta_0 \xi (S_j \mathcal{D}^j)^2 + \frac{e}{2m} \beta_0 (1 + \xi) (S_k B^k) \quad (4.11)$$

Provided the wave function  $\psi$  is an eigenstate to the Hamilton operator  $H$  with an eigenvalue  $E \neq 0$ , one more step in the reduction is possible and we can reduce (4.11)

to the first three components of  $\beta_0^2\psi$  denoted here by  $\chi$ . For simplicity we write the result of this further reduction for  $A_0 = 0$

$$E^2\chi = \left[ m^2 - \mathcal{D}^i\mathcal{D}_i - \frac{e}{m}(T_k B^k) - \frac{1}{m^2}(T_i\mathcal{D}_i)^2(\mathcal{D}^j\mathcal{D}_j) - \frac{1}{m^2}(T_k B^k)^4 - \frac{e}{m^2}(T_i\mathcal{D}^i)^2(T_k B^k) \right] \chi \quad (4.12)$$

Note that (4.11) is already of second order and as a consequence eq. (4.12) is of fourth order.

The important conclusion of the reduction process to the physical components of the wave function is that the troublesome term

$$\frac{ie}{2m}F^{\mu\rho}(\beta_\rho\beta_0\beta_\mu + \beta_\rho g_{\mu 0})$$

vanishes, i.e., it reduces to zero. This means that it does not matter whether one couples the photon via the minimal scheme in the Lagrangian or as suggested in [11] in the Hamilton form. The end result of the reduction will be the same which implies that the above term does not have any physical meaning. Actually Kemmer has discussed a similar looking term at hand of the second order equation (3.5) [5]. Indeed it is not ruled out yet that for instance term involving  $\frac{ie}{2m}F^{ij}(\beta_j\beta_k\beta_i + \beta_j g_{ik})$  are physically meaningful. This, however, presupposes that (3.5) itself is meaningful. The next two section are centered around this problematic.

## 5 The second order wave equation

To understand better the implications of the equation (3.5) it is helpful to have a glimpse at what happens in the Dirac theory. In the presence of electromagnetic field the Dirac equation reads

$$\begin{aligned} \Lambda_\pm &\equiv \gamma_\mu \mathcal{D}^\mu \pm m \\ \Lambda_- \Psi &= 0 \end{aligned} \quad (5.1)$$

The second order equation is obtained by writing  $\Lambda_+\Lambda_-\Psi = 0$  where

$$\Lambda_+\Lambda_- = \mathcal{D}^\alpha\mathcal{D}_\alpha - m^2 - \frac{e}{2}F^{\mu\nu}\sigma_{\mu\nu} \quad (5.2)$$

where as usual  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ . Now in the Dirac theory it is simple to pass from solutions of the second order wave equation  $\Psi$  to solutions of the Dirac equation. By virtue of

$$[\Lambda_+, \Lambda_-] = 0$$

it follows that

$$\Lambda_-(\Phi) = \Lambda_-(\Lambda_+\Psi) = 0 \quad (5.3)$$

In other words, if  $\Psi$  solves  $\Lambda_+ \Lambda_- \Psi = 0$  then  $\Phi = \Lambda_+ \Psi$  is a solution of (5.1). This one-to-one correspondence makes the second order Dirac equation a meaningful physical equation. Indeed its practical usefulness lies in the fact that one can easily obtain solutions of the Dirac equation for certain field configurations by using (5.2). We should therefore try to find out if the successful Dirac story can formally be taken over to the case of the second order Kemmer equation. As a first step in this direction we make an ansatz to factorize the operator  $\Omega_1(\mathcal{D})$  in the form

$$d_1(\mathcal{D})\Lambda(\mathcal{D}) = \Omega_1(\mathcal{D}) \quad (5.4)$$

where  $d_1(\mathcal{D})$  is to be found. Such an operator indeed exists and the solution is given by

$$d_1(\mathcal{D}) = \frac{1}{m} \left[ (\mathcal{D}^\alpha \mathcal{D}_\alpha) - m^2 \right] + \beta_\nu \mathcal{D}^\nu - \frac{1}{m} \beta_\sigma \beta_\delta \mathcal{D}^\delta \mathcal{D}^\sigma \quad (5.5)$$

This is a straightforward generalization of the operator of Takahashi and Umezawa [16] who have found that in the free case

$$d_1(\partial)\Lambda(\partial) = -(\square + m^2) \quad (5.6)$$

Although this looks already quite promising the fact that the commutator

$$[d_1(\mathcal{D}), \Lambda(\mathcal{D})] = \frac{e}{2m} \beta_\sigma \beta_\rho \beta_\delta (\partial^\rho F^{\delta\sigma}) - \frac{ie}{m} \beta_\sigma F^{\rho\sigma} \mathcal{D}_\rho \quad (5.7)$$

is non-zero spoils a complete analogy to the Dirac case. We therefore do not have a clear prescription for how to obtain solutions of the KDP equations out of solutions of the second order Kemmer equations (3.5). It could still be, despite (5.7), that some kind of a map similar to (5.3) exists. However, the situation is more serious once one realizes that (3.5) is not unique and belongs to a class of second order Klein-Gordon type equations. Indeed, we can multiply eq. (3.2) by some non-zero operator  $\tilde{d}(\mathcal{D})$  to arrive at

$$\tilde{d}(\mathcal{D})\Lambda(\mathcal{D}) = \mathcal{D}^\alpha \mathcal{D}_\alpha - m^2 + \mathcal{G}[A_\mu] \quad (5.8)$$

where  $\mathcal{G}[A_\mu]$  is a functional of the four-potential which vanishes in the interaction free case. An explicit example is

$$\begin{aligned} d_1(\mathcal{D}) &\rightarrow d_1(\mathcal{D}) + \frac{ie}{2m} S_{\rho\sigma} F^{\rho\sigma} \\ \Omega_1(\mathcal{D}) &\rightarrow \Omega_1(\mathcal{D}) - \frac{ie}{2m} \beta_\delta \beta_\sigma \beta_\mu F^{\sigma\delta} \mathcal{D}^\mu + \frac{ie}{2} F^{\mu\nu} S_{\mu\nu} \end{aligned} \quad (5.9)$$

Again this shifted operator will not commute with  $\Lambda(\mathcal{D})$ . We cannot in analogy to (5.9) redefine  $\Lambda_+$  in the Dirac theory without spoiling the crucial equation (5.3). A preferable operator to be constructed in the KDP theory would be one which satisfies (5.8) but also  $[\tilde{d}(\mathcal{D}), \Lambda(\mathcal{D})] = 0$ . This is, however, too strong a requirement. Note that the main objective of (5.6) in the free case is to interpret  $m$  as the mass of the

particle. Also  $d_1(\partial)$  and  $\Lambda(\partial)$  commute. If we insist on a higher order wave equation containing interaction terms we have to give up the requirement that the product of the two operators is of Klein-Gordon type (5.8). We will suggest such an equation in the next section.

## 6 The third order wave equation

Giving up the requirement in form of eq. (5.8) we are looking for an operator  $d_2(\mathcal{D})$  with the following properties

$$\begin{aligned} d_2(\mathcal{D})\Lambda(\mathcal{D})\psi &= 0 \\ [d_2(\mathcal{D}), \Lambda(\mathcal{D})] &= 0 \end{aligned} \quad (6.1)$$

We will skip the calculational details and give only the main results here. Using the algebra (1.2) and on account of  $\partial_\mu F^{\mu\nu} = 0$ , we have

$$[\beta_\mu, S_{\delta\sigma}] = g_{\mu\delta}\beta_\sigma - g_{\mu\sigma}\beta_\delta \quad (6.2)$$

as well as

$$\beta_\mu\beta_\delta\beta_\sigma(\partial^\mu F^{\delta\sigma}) = -\frac{1}{2}\beta_\sigma\beta_\mu\beta_\delta(\partial^\mu F^{\delta\sigma}) \quad (6.3)$$

It then follows that

$$\begin{aligned} \left[\frac{\mathcal{D}^\alpha\mathcal{D}_\alpha}{m}, \Lambda(\mathcal{D})\right] &= -2\frac{ie}{m}F^{\delta\rho}\beta_\rho\mathcal{D}_\delta \\ \left[\frac{ie}{2m}S_{\rho\sigma}F^{\rho\sigma}, \Lambda(\mathcal{D})\right] &= -\frac{ie}{m}F^{\rho\sigma}\beta_\sigma\mathcal{D}_\rho - \frac{e}{2m}\beta_\sigma\beta_\rho\beta_\delta(\partial^\rho F^{\delta\sigma}) \end{aligned} \quad (6.4)$$

Combining these findings with (5.7) we see immediately that the operator

$$d_2(\mathcal{D}) \equiv d_1(\mathcal{D}) + \frac{ie}{2m}S_{\delta\sigma}F^{\delta\sigma} - \frac{\mathcal{D}^\alpha\mathcal{D}_\alpha}{m} \quad (6.5)$$

indeed satisfies the desired commutation relation in (6.1). The equation to be solved reads now

$$\begin{aligned} -\Omega_2(\mathcal{D})\psi \equiv -d_2(\mathcal{D})\Lambda(\mathcal{D})\psi &= \left[ m^2 + \frac{ie}{2m}(\beta_\sigma\beta_\delta\beta_\mu - \beta_\sigma g_{\delta\mu})F^{\mu\sigma}\mathcal{D}^\delta \right. \\ &\quad \left. - \frac{ie}{2m}\beta_\mu S_{\delta\sigma}\mathcal{D}^\mu F^{\delta\sigma} + \frac{1}{m}\beta_\mu\mathcal{D}^\mu\mathcal{D}^\alpha\mathcal{D}_\alpha \right]\psi = 0 \end{aligned} \quad (6.6)$$

Essentially we have achieved our goal since  $\phi \equiv d_2(\mathcal{D})\psi$  is a solution of (3.2) provided  $\psi$  is a solution of (6.6). The last equation is quite different from (3.5). It is a third order equation which upon reduction becomes a fourth order equation. This is

understandable since the reduced eq. (4.11) is already of second order. Note also that the term  $F^{\mu\nu}S_{\mu\nu}$  vanished in (6.6).

It should be also stressed that (6.6) does not have the limit of the Klein-Gordon equation in the free case. This has consequences for the interpretation of the interaction terms in (6.6). In the free case we need eq. (5.6) to interpret  $m$  as the mass of the particle. Indeed this is the only possible interpretation in the free case. Since as we said (6.6) does not have the Klein-Gordon limit we should not expect that every term in this equation will have a simple physical interpretation when we switch on the interaction. We should regard (6.6) as an equation equivalent to (3.2) as far as the solutions for the wave functions are concerned. Due to (3.7) a part of the interaction terms can, however, be interpreted as spin-field interaction at the order  $e^2$ . This is clear after a reduction similar to (4.11) is done.

Although (6.6) is, from the mathematical point of view, the counterpart to the second order Dirac equation (this despite the fact that (6.6) is of third order) it is clear that its physical implications are quite different. In addition to the differences discussed above we should also mention that (6.6) will not be of much practical help to obtain solutions of the KDP equation. This is clear since it is not a second order Klein-Gordon type equation.

## 7 Conclusions

We have tried in this letter to shed some light on the non-trivial nature of the electromagnetic coupling in the KDP equation. To do so we have reduced the KDP equation to the physical components of the wave function. Along with the reduction process a term which resisted a physical interpretation has vanished. We could therefore show that this term does not have any physical significance. We argued that the second order wave equation proposed by Kemmer as an analogy to the second order Dirac equation has only a limited applicability (in contrast to the corresponding Dirac case) as (i) it belongs to a class of possible second order equations, (ii) it lacks the important property which would allow to go back to the solutions of (3.2). Strictly speaking, the analogy to the second order Dirac equation is then lost. In view of all these points we do not think that it is really necessary to give an interpretation of the interaction terms in every such equations. To obtain a counterpart (at least from the mathematical point of view) to the second order Dirac equation we have suggested a third order equation having the desired properties which allows one to map the solution of the latter equation to solutions of (3.2), but lacks a simple physical interpretation since it does not reduce to the Klein-Gordon equation in the free case.

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## References

- [1] E. Majorana, Nuovo Cim. **9** (1932) 335; Y. Nambu, Prog. Theor. Phys. Suppl. **37-38** (1966) 368
- [2] P. A. M. Dirac, Proc. Roy. Soc. **A322** (1971) 435; ibid **A328** (1972) 1
- [3] N. Mukunda, H. Van Dam and L. C. Biedenharn, Lecture Notes in Physics **165**, Springer Verlag, Berlin 1982
- [4] S. Weinberg, Phys. Rev. **133** (1964) B1318; D. Shay and R. H. Good, Phys. Rev. **179** (1969) 141
- [5] N. Kemmer, Proc. Roy. Soc. **A173** (1939) 91
- [6] R. J. Duffin, Phys. Rev. **54** (1938) 1114
- [7] Harish-Chandra, Proc. Roy. Soc. **A186** (1946) 502
- [8] H. Umezawa, *Quantum Field Theory*, North-Holland Publishing Company, Amsterdam 1956
- [9] P. Ghose, D. Home and M. N. Sinha Roy, Phys. Lett. **A183** (1993) 267
- [10] P. Ghose and D. Home, Phys. Lett. **A191** (1994) 362
- [11] P. Ghose, Foundation of Physics **26** (1996) 1441
- [12] P. R. Holland, *The Quantum Theory of Motion*, Cambridge University Press 1993
- [13] B. Ancochea, A. Bramon, R. Munoz-Tapia and M. Nowakowski, Phys. Lett. **B389** (1996) 149
- [14] M. Nowakowski, in preparation
- [15] V. B. Berestetskii and E. M. Lifschitz, *Quantum Electrodynamics*, Pergamon Press 1982, chapter IV
- [16] Y. Takahashi and H. Umezawa, Prog. Theor. Phys. **9** (1953) 1